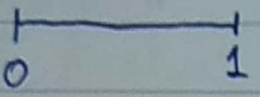
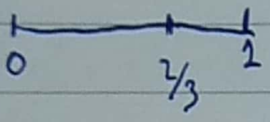
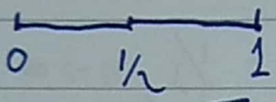
18/04/2018



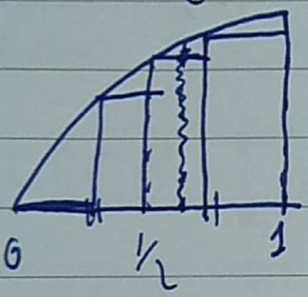
$Q=\{0,1\}$, $\|Q\|=1$



$P_1=\{0, 2/3, 1\}$, $\|P_1\|=2/3$



$P_2=\{0, 1/2, 1\}$, $\|P_2\|=1/2$



$P=\{0, 1/2, 1\}$

$$L(f,P)=0\left(\frac{1}{2}\cdot 0\right)+\frac{1}{\sqrt{2}}\left(1\cdot\frac{1}{2}\right)=\frac{1}{2\sqrt{2}}$$

Πόρισμα

Η $f:[\alpha,b]\to\mathbb{R}$ είναι Riemann ολοκληρώσιμη (φραγμένη) αν και μόνο αν $\exists (P_n)_n$ διαμερ. $: U(f,P_n)-L(f,P_n)\to 0$

Αποδ: $(\Leftarrow)$

~~Έστω/έχω.~~

Έστω ότι $\exists (P_n)_n$ αυ. διαμ. ώστε $\overbrace{U(f,P_n)-L(f,P_n)}^{\delta_n} \xrightarrow[n\to\infty]{} 0$

Θ.δ.ο. ισχύει η συνθ. του Riemann

Έστω $\varepsilon>0$ ψάχνουμε P_ε $U(f,P_\varepsilon)-L(f,P_\varepsilon)<\varepsilon$

$\delta_n\to 0 \Rightarrow \forall\varepsilon>0$, $\exists n_0\in\mathbb{N}$ ώστε $\forall n\geq n_0: \delta_n<\varepsilon$. Επιλογή: $P_\varepsilon=P_{n_0}$

$\Rightarrow U(f,P_\varepsilon)-L(f,P_\varepsilon)=U(f,P_n)-L(f,P_n)=\delta_{n_0}<\varepsilon$

Άσκ. $f:[\alpha,\beta]\to\mathbb{R}$ φραγμένη
(P_n) ακ. διαμ. διαμ.
$U(f,P_n)\to \ell$
$L(f,P_n)\to \ell$

f R-ολοκληρ. αφού $U(f,P_n)-L(f,P_n)\to \ell-\ell=0$

$\underline{\int_\alpha^\beta} f=\overline{\int_\alpha^\beta} f=\int_\alpha^\beta f\ \exists$

$L(f,P_n)\le \underline{\int_\alpha^\beta} f=\int_\alpha^\beta f=\overline{\int_\alpha^\beta} f\le U(f,P_n)\to \int_\alpha^\beta f=\ell$

Άσκ. $f[0,1]\to\mathbb{R}$ R-ολοκλ. : $f(x)=0,\ \forall x\in\mathbb{Q}\cap[0,1]$ Δ.Ο. $\int_0^1 f=0$

∃ από υποθ.

$\exists\ \underline{\int_0^1} f=\overline{\int_0^1} f=\int_0^1 f$

Έστω P διαμ. $U(f,P)\ge\overline{\int_0^1} f=\int_0^1 f$
$L(f,P)\le\underline{\int_0^1} f=\int_0^1 f$

$L(f,P)=\sum\limits_{k=0}^{n-1} m_k(x_{k+1}-x_k)\le 0$

$m_k=\inf\{f(x):x\in[x_k,x_{k+1}]\}\le 0$, $\Rightarrow \forall P$ διαμ. $L(f,P)\le 0$

όμοια: $M_k=\sup\{f(x):x\in[x_k,x_{k+1}]\}\ge 0$.

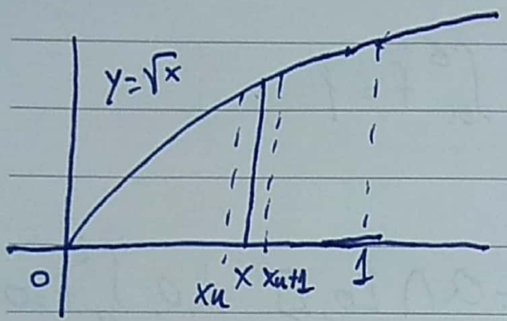
$\left.\begin{array}{l}\sup\{L(f,P):P \text{ διαμ}\}=\underline{\int_0^1}\\ L(f,P)\le 0\ \forall P\end{array}\right\}\overset{\text{ορισ}}{\underset{\sup}{\Longrightarrow}}\underline{\int_0^1} f\le 0$

Παρόμοια $U(f,P) \geq 0$,

$\forall P \Rightarrow \inf\{U(f,P) : P \text{ διαμ}\} = \overline{\int_0^1} f \geq 0 \Rightarrow \underline{\int_0^1} f = 0 \; \left(= \underline{\int_0^1} f = \overline{\int_0^1} f\right)$

$f : [0,1] \to \mathbb{R}$, $f(x) = \sqrt{x}$. Δ.ο. f. R-ολοκλ. $\int_0^1 \sqrt{x}\,dx = ;$

$$\int_0^1 x^{1/2}\,dx = \left[\frac{2}{3} x^{3/2}\right]_0^1 = \frac{2}{3}$$



$P_n = \{0 = x_0 < x_1 < \dots < x_{n-1} < x_n = 1\}$

$U(f, P_n) = \sum_{k=0}^{n-1} M_k (x_{k+1} - x_k)$

$M_k = \sup\{f(x) : x \in [x_k, x_{k+1}]\} = \sup[f(x_k), f(x_{k+1})] = f(x_{k+1})$

$$U(f, P_n) = \sum_{k=0}^{n-1} f(x_{k+1})(x_{k+1} - x_k)$$

$$L(f, P_n) = \sum_{k=0}^{n-1} f(x_k)(x_{k+1} - x_k)$$

$f(x_k) = \sqrt{x_k}$, $x_k = \frac{k^2}{n^2}$, $k = 0, 1, \dots, n-1, n$

$$P_n = \left\{x_0 = 0 < \frac{1}{n^2} = x_1 < x_2 = \frac{2^2}{n^2} < \dots < x_{n-1} = \frac{(n-1)^2}{n^2} < x_n = \frac{n^2}{n^2} = 1\right\}$$

$$U(f, P_n) = \sum_{k=0}^{n-1} \frac{k+1}{n}\left(\frac{(k+1)^2}{n^2} - \frac{k^2}{n^2}\right) \to 2/3$$

$$L(f, P) = \sum_{k=0}^{n-1} \frac{k}{n}\left(\frac{(k+1)^2}{n^2} - \frac{k^2}{n^2}\right) \to 2/3$$

$$U(f,P_n) = \frac{1}{n^3}\sum_{k=0}^{n-1}(k+1)(2k+1)$$

$$\frac{1}{n^3}\left(\sum_{k=0}^{n-1}(2k^2+3k+1)\right) = \frac{1}{n^3}\sum_{k=0}^{n-1}2k^2 + \underbrace{\frac{1}{n^3}\sum_{k=0}^{n-1}3k}_{\to 0} + \cancel{\frac{1}{n^3}\,n} \to 0$$

γενικά $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ $\qquad \frac{2\cdot 2}{6} = \frac{4}{6} = \frac{2}{3}$ //

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$\Rightarrow f$ R-ολοκλ. και $\int_0^1 f = 2/3$.

Άσκ. $f[a,b] \to \mathbb{R}$ μονότονη (φραγμένη) $\Rightarrow f$: R-ολοκλ.

χωρίς βλάβη της γενικότητας υποθ. $f \uparrow$

$P_n = \{x_0 = a < x_1 = a + \frac{b-a}{n} < x_2 = a + 2\frac{(b-a)}{n} < \dots < x_k = a + k\frac{(b-a)}{n} < \dots < x_n =$
$a + \frac{n(a-b)}{n} = b$.

$$U(f,P_n) = \sum_{k=0}^{n-1} M_k(x_{k+1}-x_k) = \sum_{k=0}^{n-1} f(x_{k+1})\left(\frac{b-a}{n}\right) = \frac{b-a}{n}\sum_{k=1}^{n} f(x_k)$$

$$L(f,P_n) = \sum_{k=0}^{n-1} m_k(x_{k+1}-x_k) = \sum_{k=0}^{n-1} f(x_k)\cdot\left(\frac{b-a}{n}\right) = \frac{b-a}{n}\sum_{k=0}^{n-1} f(x_k).$$

$$U(f,P_n) - L(f,P_n) = \frac{b-a}{n}\left\{\left[\sum_{k=1}^{n-1} f(x_k) + f(x_n)\right] - \left[\sum_{k=1}^{n-1} f(x_k) + f(x_0)\right]\right\}$$

$$= \frac{b-a}{n}\left[f(b) - f(a)\right] \to 0, \quad n \to \infty$$

Άσκ. f = σταθερή. Έστω $f(x) = c$, $\forall x \in [a,b]$

$$\underset{\substack{\parallel \\ c(b-a)}}{U(f,P)} = \underset{\substack{\parallel \\ c(b-a)}}{L(f,P)} \quad \forall P$$

Υποθ. ότι $\forall P$ διαμερ. του $[a,b]$

$U(f,P) = L(f,P) \overset{\Delta.\varsigma.}{\Rightarrow} f$ = σταθ.

$f : [a,b] \to \mathbb{R}$ φραγμ.

Διαλέγουμε $P = \{a,b\}$

$$\left.\begin{aligned} U(f,P) &= \sup\{f(x) : x \in [a,b]\}(b-a) \\ L(f,P) &= \inf\{f(x), x \in [a,b]\}\ (b-a) \end{aligned}\right\} \quad \begin{aligned} &M(b-a) = m(b-a) \\ &\Rightarrow M = m \end{aligned}$$

Άρα f σταθ. διότι αν $\exists\ x, y \in [a,b]$: $m \le f(x) \le f(y) \le M$ ⇒

$m = M$ Άρα σταθ.